# Diophantine Approximations

Nikhil Shah* Asher Shmidov* Abigail Qiu*
April Zhan*
Ananya Prasanna

August 2nd, 2025

*Equal contribution.

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# §1 Introduction

This paper begins by considering the following question: "if you were in the euclidean plane at (0, 0), and at every lattice point there exists a star, how far could you see?" We considered this question and its natural extensions, such as starting at any arbitrary point, and and changing the radii of the stars. Then, we consider rational approximations of irrational numbers, and their intrinsic relationship with convergents of continued fractions, proving that All convergents are "good approximations" for an irrational. Then, we consider the circumstances in which continued fractions are periodic, before ending with a variety of observations and conjectures surrounding the topic of continued fractions.

# §2 Stargazing

Consider an observer in $\mathbb{R}^2$ at the point $(0, 0)$, where the observer may "see" along a line of sight, and every lattice point becomes a 0-radius star.

**Question 2.1.** Can the observer ever see "to infinity," or will every line of sight fall upon a star

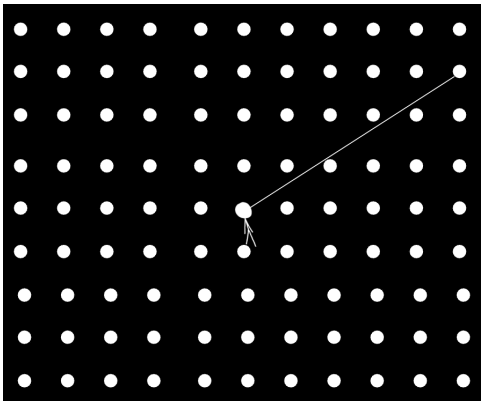


Figure 1: The observer, embedded into $\mathbb{R}^2$, enjoying a night sky

**Proposition 2.2**

The observer sees a star if and only if their line of sight has rational slope

*Proof.* For a given star at $(x, y)$, $x, y \in \mathbb{Z}$ the star falls upon the line $y = \frac{y}{x}x$ trivially, and that line cross the origin, so the observer will see that star if they look at slope $\frac{y}{x}$, which has rational slope. Conversely, if the observer is looking at any slope $\frac{p}{q}$, all stars at coordinates $(kp, kq)$, for $k \in \mathbb{N}$ will be seen, since $kq = \frac{q}{p}kp$. □

Thus, while the observer can see an infinite number of stars in an infinite number of directions, there are also an infinite number of directions in which they will see all the way to infinity.

## §2.1 Non-Zero Radii Stars

Now, consider stars having radii. That is let lattice points be surrounded by a small circular disk of radius $\epsilon \in \mathbb{R}$.

We claim that the line of vision will always fall in the radius of light surrounding some star. In other words, you will not see darkness all the way "to infinity".

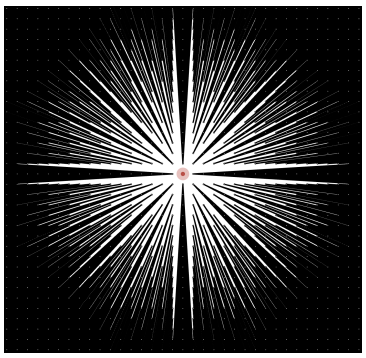


Figure 2: Example of finite vision with non-zero radii

**Theorem 2.3**

For all $\lambda, \epsilon \in \mathbb{R}$ where $\lambda$ is irrational and $\epsilon > 0$, there exists a nonzero integer $x$ such that $\lambda x - \lfloor \lambda x \rfloor \leq \epsilon$. (For any $a \in \mathbb{R}$, we denote $a - \lfloor a \rfloor$ as $\{a\}$).

*Proof.* Let $\lambda, \epsilon \in \mathbb{R}$ where $\lambda$ is irrational and $\epsilon > 0$. Recall that $0 \leq \{a\} < 1$ for all $a \in \mathbb{R}$.

So if $\epsilon \geq 1$, then $\{\lambda x\} < \epsilon$ for all $x \in \mathbb{Z}$. Now we consider $\epsilon < 1$.

We want to show that there exist distinct integers $x_1, x_2$ such that

$$0 \leq \{\lambda \cdot x_1\} - \{\lambda \cdot x_2\} \leq \epsilon$$

We can then let $x_3 = x_1 - x_2$. $\mathbb{Z} \ni x_3 \neq 0$. We can show that $\{\lambda \cdot x_3\} \leq \epsilon$, as

$$\begin{aligned} \lambda x_3 &= \lambda \cdot (x_1 - x_2) \\ &= \lambda \cdot x_1 - \lambda \cdot x_2 \\ &= (\lfloor \lambda \cdot x_1 \rfloor + \{\lambda \cdot x_1\}) - (\lfloor \lambda \cdot x_2 \rfloor + \{\lambda \cdot x_2\}) \\ &= (\lfloor \lambda \cdot x_1 \rfloor - \lfloor \lambda \cdot x_2 \rfloor) + (\{\lambda \cdot x_1\} - \{\lambda \cdot x_2\}) \end{aligned}$$

If such $x_1, x_2$ exist, then

$$0 \leq \{\lambda \cdot x_1\} - \{\lambda \cdot x_2\} \leq \epsilon$$

so $0 \leq \{\lambda x_1\} - \{\lambda x_2\} < 1$. Additionally, $\lfloor \lambda \cdot x_1 \rfloor - \lfloor \lambda \cdot x_2 \rfloor \in \mathbb{Z}$.

Hence, the greatest integer less than or equal to

$$(\lfloor \lambda \cdot x_1 \rfloor - \lfloor \lambda \cdot x_2 \rfloor) + (\{\lambda \cdot x_1\} - \{\lambda \cdot x_2\})$$

is $\lfloor \lambda \cdot x_1 \rfloor - \lfloor \lambda \cdot x_2 \rfloor$.
In other words,

$$\lfloor \lambda \cdot (x_1 - x_2) \rfloor = \lfloor \lambda \cdot x_1 \rfloor - \lfloor \lambda \cdot x_2 \rfloor$$

Hence,

$$\begin{aligned} \lfloor \lambda x_3 \rfloor &= \lfloor \lambda x_1 \rfloor - \lfloor \lambda x_2 \rfloor \\ \implies \{\lambda x_3\} &= \lambda x_3 - \lfloor \lambda x_3 \rfloor = \lambda x_3 - (\lfloor \lambda x_1 \rfloor - \lfloor \lambda x_2 \rfloor) \\ &= \lambda(x_1 - x_2) - (\lfloor \lambda x_1 \rfloor - \lfloor \lambda x_2 \rfloor) \\ &= \lambda x_1 - \lambda x_2 - (\lfloor \lambda x_1 \rfloor - \lfloor \lambda x_2 \rfloor) \\ &= (\lfloor \lambda x_1 \rfloor + \{\lambda x_1\}) - (\lfloor \lambda x_2 \rfloor + \{\lambda x_2\}) - (\lfloor \lambda x_1 \rfloor - \lfloor \lambda x_2 \rfloor) \\ &= \{\lambda x_1\} - \{\lambda x_2\} \end{aligned}$$

Hence,

$$\begin{aligned} 0 \leq \{\lambda x_1\} - \{\lambda x_2\} = \{\lambda x_3\} &\leq \epsilon \\ \implies 0 \leq \{\lambda x_3\} &\leq \epsilon \end{aligned}$$

which shows the existence of the desired integer.

Now, we will show that such $x_1$ and $x_2$ exist.

Let $q = \lfloor \frac{1}{\epsilon} \rfloor$, so

$$1 = q \cdot \epsilon + r,$$

where $0 \leq r < \epsilon$.

Now, divide the interval $[0, 1)$ on the real number line into $q + 1$ portions, of which $q$ have length $\epsilon$ and one has length $r$.

Let $x_1, x_2, \ldots, x_{q+1}$ be distinct integers. We plot $\{\lambda x_i\}$ for each $i \in \{1, 2, \ldots, q + 1\}$ on the number line. The number line consists of $q$ $\epsilon$-length intervals. So by the Pigeonhole Principle, at least two of the values of $\{\lambda x_i\}$ must lie in the same $\epsilon$-length interval and thus have a maximum difference of $\epsilon$.

Denote two such points as $\{\lambda x_j\}$ and $\{\lambda x_k\}$. WLOG, let

$$\{\lambda x_k\} \geq \{\lambda x_j\}$$

so that

$$0 \leq \{\lambda x_k\} - \{\lambda x_j\} \leq \epsilon$$

And by the reasoning previously established, $\{\lambda(x_k - x_j)\} \leq \epsilon$.

Hence, the nonzero integer $x_k - x_j$ satisfies the given conditions. □

**Corollary 2.4**

For all $\epsilon, \lambda \in \mathbb{R}$ with $\epsilon > 0$ and irrational $\lambda$, there exists a lattice point $(a, b)$ such that $(a, b) \neq (0, 0)$ and the distance between $(a, b)$ and a point on the line $y = \lambda x$ is less than or equal to $\epsilon$.

*Proof.* Let line $\ell$ be defined by $y = \lambda x$, where $\lambda \in \mathbb{R}$ and is irrational. Let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. By 2.3, there exists a nonzero $x_0 \in \mathbb{Z}$ such that

$$\lambda x_0 - \lfloor \lambda x_0 \rfloor \leq \epsilon$$

Hence, the lattice point $(x_0, \lfloor \lambda x_0 \rfloor)$ and the point $(x_0, \lambda x_0)$ on line $\ell$ are within a distance of $\epsilon$ from each other. □

## §2.2 Non-Origin Vantage Point

Dissatisfied with being unable to see to infinity, our observer now moves to some arbitrary point within the coordinate plane. Considering stars still having radius, suppose our new line of sight is defined by $y = \lambda x + b$, for some, not necessarily rational $\lambda$.
First, consider a rational slope. In this case, we claim that our observer may not be able to see arbitrarily close to a lattice point. In other words, if the radius around each star is sufficiently small, our observer can see to infinity without hitting the circle around a star with their line of vision.

**Proposition 2.5**

Given a line $\ell$ that passes through $(x_0, y_0)$ with rational slope $\frac{p}{q}$, it is not guaranteed that there exists a point arbitrarily close to $\ell$.

*Proof.* Let $x_0, y_0 \in \mathbb{R}$ and $p, q \in \mathbb{Z}, q \neq 0$, and $\gcd(p, q) = 1$ so that $\frac{p}{q}$ is a fully simplified rational. Let $\ell$ be a line with slope $\frac{p}{q}$ that passes through $(x_0, y_0)$.
So $\ell$ is defined by $y - y_0 = \frac{p}{q}(x - x_0)$. Rearrange to get

$$0 = px - qy + (qy_0 - px_0)$$

Let $a, b \in \mathbb{Z}$ so that $(a, b)$ is a lattice point. By the distance formula from a point to a line, the distance between $(a, b)$ to $\ell$ is

$$\frac{|pa - qb + (qy_0 - px_0)|}{\sqrt{p^2 + q^2}}$$

Since $p, q, a, b \in \mathbb{Z}$, we have $pa - qb \in \mathbb{Z} \implies -(pa - qb) \in \mathbb{Z}$.

If $qy_0 - px_0$ is not an integer, then $-(pa - qb) \neq qy_0 - px_0$, which means

$$pa - qb + (qy_0 - px_0) \neq 0$$

In this case, we have $qy_0 - px_0 = \lfloor qy_0 - px_0 \rfloor + \{qy_0 - px_0\}$ where $\{qy_0 - px_0\}$ is fixed and greater than 0.

And the closest two integers to $qy_0 - px_0$ are $\lfloor py_0 - qx_0 \rfloor$ and $\lceil py_0 - qx_0 \rceil$. So the minimum absolute difference between an integer and $py_0 - qx_0$ is

$$\min(\{qy_0 - px_0\}, 1 - \{qy_0 - px_0\})$$

which we will call $m$.

Hence,

$$\frac{|pa - qb + (qy_0 - px_0)|}{\sqrt{p^2 + q^2}} \geq \frac{m}{\sqrt{p^2 + q^2}}$$

Hence, the distance between $\ell$ and a lattice point cannot be arbitrarily small.

Thus, the minimum distance between a lattice point and $y - y_0 = \frac{p}{q}(x - x_0)$ is dependent on $x_0, y_0, p$, and $q$, and is not necessarily arbitrarily small. Thus, there exist rational slopes and $\epsilon$ values such that the observer may see to infinity.

□

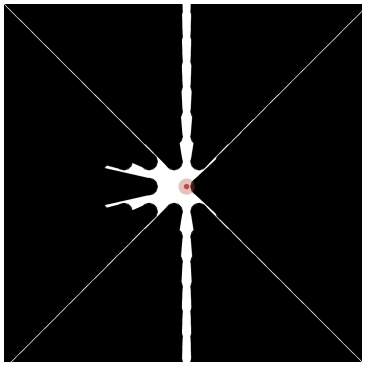


Figure 3: Observer at (0.5, 0), able to see to infinite with a vertical line of sight, or slope ±1

Now, consider irrational $\lambda$

**Lemma 2.6**

For all $a, b \in \mathbb{R}$, if $\{a\} + \{b\} < 1$, then $\{a + b\} = \{a\} + \{b\}$. If $\{a\} + \{b\} \geq 1$, then $\{a+b\} = \{a\} + \{b\} - 1$. Here, $\{n\}$ denotes $n - \lfloor n \rfloor$, where $\lfloor n \rfloor$ is the greatest integer less than or equal to $n$ for all $n \in \mathbb{R}$.

*Proof.* Let $n \in \mathbb{R}$. We first show that $0 \leq \{n\} = n - \lfloor n \rfloor < 1$.
By definition, $\lfloor n \rfloor \leq n$, which gives us $0 \leq n - \lfloor n \rfloor$.
Now suppose for the sake of contradiction that $n - \lfloor n \rfloor \geq 1$. Then $n \geq \lfloor n \rfloor + 1$. Since $\lfloor n \rfloor + 1 > \lfloor n \rfloor$, $\lfloor n \rfloor$ would not the be greatest integer less than or equal to $n$, which contradicts the definition of $\lfloor n \rfloor$. Hence, $n - \lfloor n \rfloor$ must be less than 1.
Thus, $0 \leq n - \lfloor n \rfloor < 1$.

Now let $a, b, \in \mathbb{R}$ and $\{a\} + \{b\} < 1$. Then $a + b = \lfloor a \rfloor + \{a\} + \lfloor b \rfloor + \{b\} < \lfloor a \rfloor + \lfloor b \rfloor + 1$. Since $0 \leq \{a\}, \{b\}$, we have $0 \leq \{a\} + \{b\}$ . So $\lfloor a \rfloor + \lfloor b \rfloor \leq a + b = \lfloor a \rfloor + \{a\} + \lfloor b \rfloor + \{b\} < \lfloor a \rfloor + \lfloor b \rfloor + 1$.

Hence, the greatest integer less than or equal to $a + b$ is $\lfloor a \rfloor + \lfloor b \rfloor$. So $\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor$. Hence,

$$\{a + b\} = (a + b) - \lfloor a + b \rfloor = (\lfloor a \rfloor + \{a\} + \lfloor b \rfloor + \{b\}) - (\lfloor a \rfloor + \lfloor b \rfloor) = \{a\} + \{b\}$$

Now, consider the case $\{a\} + \{b\} \geq 1$. Since $\{a\}, \{b\} < 1$, $\{a\} + \{b\} < 2$. So $1 \leq \{a\} + \{b\} < 2$.
Then, $\lfloor a \rfloor + \lfloor b \rfloor + 1 \leq a + b = \lfloor a \rfloor + \{a\} + \lfloor b \rfloor + \{b\} < \lfloor a \rfloor + \lfloor b \rfloor + 2$.

Hence, the greatest integer less than or equal to $a+b$ is $\lfloor a \rfloor + \lfloor b \rfloor + 1$. So $\lfloor a+b \rfloor = \lfloor a \rfloor + \lfloor b \rfloor + 1$. Hence,

$$\{a+b\} = (a+b) - \lfloor a+b \rfloor = (\lfloor a \rfloor + \{a\} + \lfloor b \rfloor + \{b\}) - (\lfloor a \rfloor + \lfloor b \rfloor + 1) = \{a\} + \{b\} - 1$$

□

**Theorem 2.7**

For all $\lambda, \epsilon, b \in \mathbb{R}$ where $\lambda$ is irrational and $\epsilon > 0$, there exists a nonzero integer $x$ such that $\{\lambda x + b\} \leq \epsilon$.

*Proof.* Let $\lambda, \epsilon, b \in \mathbb{R}$ where $\lambda$ is irrational and $\epsilon > 0$. Recall that $0 \leq \{a\} < 1$ for all $a \in \mathbb{R}$.

So if $\epsilon \geq 1$, then $\{\lambda x + b\} < \epsilon$ for all $x \in \mathbb{Z}$. Now we consider $\epsilon < 1$.

Consider the case $\{b\} < \epsilon$. Then $0 < \epsilon - \{b\}$.
By Theorem 1.3, there exists a nonzero $x \in \mathbb{Z}$ such that

$$\begin{aligned} \lambda x - \lfloor \lambda x \rfloor &\leq \epsilon - \{b\} \\ \implies \{\lambda x\} &\leq \epsilon - \{b\} \end{aligned}$$

Hence,

$$\{\lambda x\} + \{b\} \leq \epsilon$$

Since $\epsilon < 1$, we have

$$\{\lambda x\} + \{b\} < 1$$

By 2.6,

$$\{\lambda x + b\} = \{\lambda x\} + \{b\}$$

Hence,

$$\{\lambda x + b\} \leq \epsilon$$

Thus, a there exists a nonzero integer $x$ that satisfies our conditions when $\{b\} < \epsilon$.

Now, we consider the case $\{b\} \geq \epsilon$.
We want to show that there exists a nonzero integer $x$ such that

$$1 \leq \{\lambda x\} + \{b\} \leq 1 + \epsilon$$

so that we can have

$$\{\lambda x + b\} = \{\lambda x\} + \{b\} - 1$$

by 2.6.

We show the existence of such $x$ as follows:

We want for

$$1 \leq \{\lambda x\} + \{b\} \leq 1 + \epsilon \iff 1 - \{b\} \leq \{\lambda x\} \leq 1 + \epsilon - \{b\}$$

By 2.3, there exists a nonzero integer $x_0$ such that

$$\{\lambda x_0\} \leq \epsilon$$

We can verify that $\{\lambda x_0\} \leq 1 + \epsilon - \{b\}$:

$$\begin{aligned} \{b\} < 1 \implies 1 + \epsilon - \{b\} &> 1 + \epsilon - 1 = \epsilon \\ \{\lambda x_0\} \leq \epsilon &< 1 + \epsilon - \{b\} \end{aligned}$$

If $1 - \{b\} \leq \{\lambda x_0\}$, then

$$\begin{aligned} 1 - \{b\} \leq \{\lambda x_0\} &\leq 1 + \epsilon - \{b\} \\ \implies 1 \leq \{\lambda x_0\} + \{b\} &\leq 1 + \epsilon \\ \implies 1 - 1 \leq \{\lambda x_0\} + \{b\} - 1 = \{\lambda x_0 + b\} &\leq 1 + \epsilon - 1 \\ \implies 0 \leq \{\lambda x_0 + b\} &\leq \epsilon \end{aligned}$$

Hence, $x_0$ satisfies our desired conditions.

Now we consider the other case, $1 - \{b\} > \{\lambda x_0\}$:

We first show that $\{\lambda x_0\} \neq 0$: Suppose for the sake of contradiction that $\{\lambda x_0\} = 0$

$$\begin{aligned} &\implies \lambda x_0 - \lfloor \lambda x_0 \rfloor = 0 \\ &\implies \lambda x_0 = \lfloor \lambda x_0 \rfloor \end{aligned}$$

Since $x_0$ is nonzero,

$$\lambda = \frac{\lfloor \lambda x_0 \rfloor}{x_0}$$

which means that $\lambda$ can be written as $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, which by definition contradicts the condition of $\lambda$ being irrational.
Hence, $\{\lambda x_0\} \neq 0 \implies 0 < \{\lambda x_0\} \leq \epsilon$
($0 \not> \{\lambda x_0\}$, as $\{\lambda x_0\} = \lambda x_0 - \lfloor \lambda x_0 \rfloor$, and $\lfloor \lambda x_0 \rfloor$ is by definition less than or equal to $\lambda x_0$).
Since $0 < \{\lambda x_0\}$, and $\{b\} < 1 \implies 0 < 1 - \{b\}$, there exists some $k \in \mathbb{N}$ such that $k\{\lambda x_0\} \geq 1 - \{b\}$. And by the Well-ordering Principle, there exists a least element in the set of all such $k$, which we will denote as $S$. Let $k_0$ be this least element.
Since $1\{\lambda x_0\} < 1\{b\}$, we must have $k_0 > 1 \implies k_0 \geq 2$.

We will show that $k_0\{\lambda x_0\} \leq 1 - \{b\} + \epsilon$.
Suppose for the sake of contradiction that $k_0\{\lambda x_0\} > 1 - \{b\} + \epsilon$

$$\begin{aligned} &\implies k_0\{\lambda x_0\} - \{\lambda x_0\} > 1 - \{b\} + \epsilon - \{\lambda x_0\} \\ &\implies (k_0 - 1)\{\lambda x_0\} > 1 - \{b\} + \epsilon - \{\lambda x_0\} \end{aligned}$$

Since $\{\lambda x_0\} \leq \epsilon$,

$$\begin{aligned} 1 - \{b\} + \epsilon - \{\lambda x_0\} &\geq 1 - \{b\} + \epsilon - \epsilon \\ \implies 1 - \{b\} + \epsilon - \{\lambda x_0\} &\geq 1 - \{b\} \\ \implies (k_0 - 1)\{\lambda x_0\} > 1 - \{b\} + \epsilon - \{\lambda x_0\} &\geq 1 - \{b\} \\ \implies (k_0 - 1)\{\lambda x_0\} &\geq 1 - \{b\} \end{aligned}$$

so $k_0 - 1$ is also satisfies the conditions necessary for a natural number to be in S. Since $\mathbb{Z} \ni k_0 \geq 2$, we have $\mathbb{Z} \ni k_0 - 1 \geq 1 \implies k_0 \in \mathbb{N}$.
Hence, $k_0 - 1 \in S$, which contradicts $k_0$ being the least element of S.
Hence, $1 - \{b\} \leq k_0\{\lambda x_0\} \leq 1 - \{b\} + \epsilon$.

Since $\epsilon \leq \{b\}$,

$$\begin{aligned} 1 - \{b\} + \epsilon &\leq 1 - \epsilon + \epsilon = 1 \\ \implies k_0\{\lambda x_0\} &\leq 1 \end{aligned}$$

We will now show that $k_0\{\lambda x_0\} \neq 1$.
Suppose for the sake of contradiction that $k_0\{\lambda x_0\} = 1$. Since $k_0 \geq 2 \implies k_0 \neq 0$,

$$\begin{aligned} \{\lambda x_0\} &= \frac{1}{k_0} \\ \implies \lambda x_0 - \lfloor \lambda x_0 \rfloor &= \frac{1}{k_0} \\ \implies \lambda x_0 = \frac{1}{k_0} + \lfloor \lambda x_0 \rfloor &= \frac{k_0 \lfloor \lambda x_0 \rfloor + 1}{k_0} \end{aligned}$$

And since $x \neq 0$,

$$\lambda = \frac{k_0 \lfloor \lambda x_0 \rfloor + 1}{k_0 x_0}$$

So $\lambda$ can be written as $\frac{p}{q}$ for some $p, q \in \mathbb{Z}$, which contradicts $\lambda$ being irrational.

Hence, $k_0\{\lambda x_0\} \neq 1 \implies k_0\{\lambda x_0\} < 1$.

We will now show that $\{\lambda(k_0 x_0)\} = k_0\{\lambda x_0\}$.

$$\begin{aligned} k_0(\lambda x_0) &= k_0(\lfloor \lambda x_0 \rfloor + \{\lambda x_0\}) \\ &= k_0 \lfloor \lambda x_0 \rfloor + k_0\{\lambda x_0\} \end{aligned}$$

Since $0 < 1 - \{b\} \leq k_0\{\lambda x_0\} < 1 \implies 0 \leq k_0\{\lambda x_0\} < 1$,
the greatest integer less than or equal to $k_0(\lambda x_0)$ must be $k_0 \lfloor \lambda x_0 \rfloor$.
In other words, $\lfloor k_0(\lambda x_0) \rfloor = k_0 \lfloor \lambda x_0 \rfloor$. It follows that

$$\begin{aligned} \{k_0(\lambda x_0)\} &= k_0(\lambda x_0) - \lfloor k_0(\lambda x_0) \rfloor \\ &= k_0(\lambda x_0) - k_0 \lfloor \lambda x_0 \rfloor \end{aligned}$$

$$= k_0(\lfloor \lambda x_0 \rfloor + \{\lambda x_0\}) - k\lfloor \lambda x_0 \rfloor$$
$$= k_0\{\lambda x_0\}$$

So $\{k_0(\lambda x_0)\} = k_0\{\lambda x_0\} \implies \{\lambda(k_0 x_0)\} = k_0\{\lambda x_0\}$.

Hence, $1 - \{b\} \leq k_0\{\lambda x_0\} = \{\lambda(k_0 x_0)\} \leq 1 - \{b\} + \epsilon$.

So

$$1 \leq \{\lambda(x_0 k_0)\} + \{b\} \leq 1 + \epsilon$$

And by 2.6,

$$1 - 1 \leq \{\lambda(k_0 x_0) + b\} \leq 1 + \epsilon - 1$$
$$\implies 0 \leq \{\lambda(k_0 x_0) + b\} \leq \epsilon$$

Since $\mathbb{Z} \ni k_0, x_0 \neq 0$, we have $\mathbb{Z} \ni k_0 x_0 \neq 0$.

Therefore, there exists a nonzero integer (namely $k_0 x_0$) that satisfies our desired conditions.

□

**Corollary 2.8**

Given $\lambda, \epsilon, x_1, y_1 \in \mathbb{R}$ with $\epsilon > 0$ and $\lambda$ irrational, there exists some lattice point with a distance of less than or equal to $\epsilon$ from a point on the line $y - y_1 = \lambda(x - x_1)$.

*Proof.* Let $\lambda, \epsilon, x_1, y_1 \in \mathbb{R}$ with $\epsilon > 0$ and $\lambda$ irrational. Let line $\ell$ be defined by $y - y_1 = \lambda(x - x_1)$.
We can rearrange to get

$$y = \lambda x - \lambda x_1 + y_1$$

By 2.7, there exists an integer $x_0$ such that $\{\lambda x_0 + (-\lambda x_1 + y_1)\} \leq \epsilon$.

Hence, the lattice point $(x_0, \lfloor \lambda x_0 + (-\lambda x_1 + y_1) \rfloor)$ is within a distance fo $\epsilon$ away from $(x_0, \lambda x_0 + (-\lambda x_1 + y_1))$, which is a point on $\ell$. □

If our line has an irrational slope but starts at some lattice point $(a, b)$, then this line has a distance of 0 from $(a, b)$. In this case, we will call $(a, b)$ a *trivial point*.

**Proposition 2.9**

Let $\lambda, \epsilon \in \mathbb{R}$, with $\epsilon > 0$ and $\lambda$ irrational. Let $\ell$ be a line with slope $\lambda$ originating from the lattice point $(x_1, y_1)$. Then there exists a non-trivial lattice point with a distance less than or equal to $\epsilon$ away from $\ell$.

*Proof.* Let $x_1, y_1 \in \mathbb{Z}$ and $\lambda, \epsilon \in \mathbb{R}$, with $\epsilon > 0$ and $\lambda$ irrational. Let $\ell$ be defined by $y - y_1 = \lambda(x - x_1)$, which has a slope of $\lambda$ and passes through the point $(x_1, y_1)$.

We can rearrange to get

$$y = \lambda(x - x_1) + y_1$$

By 2.7, there exists a nonzero integer $k$ such that

$$\{\lambda k + y_1\} \leq \epsilon$$

We can let $x_0 = x_1 + k$. Since $k \neq 0$, we have $x_0 \neq x_1$.
Since $k, x_1 \in \mathbb{Z}$, we have $x_1 + k = x_0 \in \mathbb{Z}$.

Hence, $(x_0, \lfloor \lambda(x_0 - x_1) + y_1 \rfloor) = (x_0, \lfloor \lambda k + y_1 \rfloor)$ has a distance of $\{\lambda k + y_1\} \leq \epsilon$ from the point $(x_0, \lambda(x_0 - x_1) + y_1)$ on line $\ell$.

Since $x_0 \neq x_1$, $(x_0, \lfloor \lambda(x_0 - x_1) + y_1 \rfloor)$ is not a trivial point.

□

We've established earlier that any line with an irrational slope can become arbitrarily close to a lattice point. Consider the line starting from the origin. How far can you see before a lattice point, excluding the origin itself, falls within the desired distance from your line of vision? Can this distance become arbitrarily large, or is it bounded depending on the radius of each "star"?

**Proposition 2.10**

Given line $\ell$ with an irrational slope $\lambda$ that passes through the origin, the maximum distance that $\ell$ reaches before being within a distance of $\epsilon$ from some lattice point is $\sqrt{\lambda^2 + 1}$.

*Proof.* Let $\lambda, \epsilon \in \mathbb{R}$ with $\lambda$ irrational and $\epsilon > 0$. Let line $\ell$ be defined by $y = \lambda x$. Let $q = \lfloor \frac{1}{\epsilon} \rfloor$.

If $\epsilon \geq 1$, then at $(0, 0)$, the distance between $\ell$ and $(0, 1)$ is less than or equal to $\epsilon$ (the same is true for the distance between $(0, 0)$ on $\ell$ and $(1, 0)$). Hence, there immediately exists a non-origin lattice point that satisfies our conditions. So now we consider $\epsilon < 1$.

We want to show that there exists some nonzero integer $x_0$ with $|x_0| \leq q$ such that $\{\lambda x_0\} \leq \epsilon$

Like before, consider the real number line in the interval $[0, 1)$. We can divide this number line into at most $q = \lfloor \frac{1}{\epsilon} \rfloor$ portions with length $\epsilon$.

Now, we plot $\{\lambda\}, \{2\lambda\}, \{3\lambda\}, \ldots, \{(q+1)\lambda\}$ onto the number line. Since the number line only consists of $q$ $\epsilon$-length intervals, at least two of $\{\lambda\}, \{2\lambda\}, \{3\lambda\}, \ldots, \{(q+1)\lambda\}$ must fall within the same $\epsilon$-length interval.

Let them be $\{a\lambda\}$ and $\{b\lambda\}$, and WLOG let $\{a\lambda\} \geq \{b\lambda\}$ so that

$$0 \leq \{a\lambda\} - \{b\lambda\} \leq \epsilon$$

which gives us

$$\begin{aligned} \lambda(a - b) &= a\lambda - b\lambda \\ &= (\lfloor a\lambda \rfloor + \{a\lambda\}) - (\lfloor b\lambda \rfloor + \{b\lambda\}) \end{aligned}$$

$$= (\lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor) + (\{a\lambda\} - \{b\lambda\})$$

Since $(\lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor) \in \mathbb{Z}$ and $0 \leq \{a\lambda\} - \{b\lambda\} \leq \epsilon < 1$, the greatest integer less than or equal to $(\lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor) + (\{a\lambda\} - \{b\lambda\})$ is $(\lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor)$.

In other words, $\lfloor (\lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor) + (\{a\lambda\} - \{b\lambda\}) \rfloor = \lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor$, so

$$\lfloor \lambda(a-b) \rfloor = \lfloor a\lambda \rfloor - \lfloor b\lambda \rfloor$$

so

$$\{\lambda(a-b)\} = \{a\lambda\} - \{b\lambda\}$$

Hence,

$$0 \leq \{\lambda(a-b)\} \leq \epsilon$$

Since $a$ and $b$ are distinct, $a - b \neq 0$. And since $a, b \in \{1, 2, \ldots, q+1\}$, the maximum value of $|a-b|$ is $|(q+1)-1| = |1-(q+1)| = q$.

Let $c = a - b$. Then $(c, c\lambda)$ is within a distance of $\epsilon$ from $(c, \lfloor c\lambda \rfloor)$, and $0 < |c| \leq q$.

The distance between $(0,0)$ and $(c, c\lambda)$ (i.e., the distance that $\ell$ can reach past the origin before coming within a distance of $\epsilon$ from a lattice point), is

$$\begin{aligned}
&\sqrt{(c-0)^2 + (c\lambda - 0)^2} \\
&= \sqrt{c^2 + c^2\lambda^2} \\
&= |c|\sqrt{1+\lambda^2} \\
&\leq q\sqrt{1+\lambda^2}
\end{aligned}$$

□

# §3 Good Approximations

The prior question of seeing to infinity can be rephrased as: "for all irrationals, does there exist some rational that is arbitrarily close?" This then provides a motivation for trying to measure how good of approximations given rationals are for irrationals.
What is a "good approximation" $\frac{p}{q}$ for a given irrational $\alpha$?

**Example 3.1**

Consider $\pi$. $\frac{314}{100} = 3.14$ approximates pi to the hundreds place. However, $\frac{22}{7} \approx 3.142$ also does, and it's intuitively "less complex," so it ought to be considered "better." Also, $\frac{355}{113} \approx 3.1415929$, while seeming about as complex as $\frac{314}{100}$, is significantly more precise an approximation.

**Definition 3.2.** A fraction $\frac{p}{q}$ is a "good approximation" to irrational $\alpha$ if and only if $|\frac{p}{q} - \alpha| \leq |\frac{a}{b} - \alpha|$, for all $a, b$ where $b \leq q$

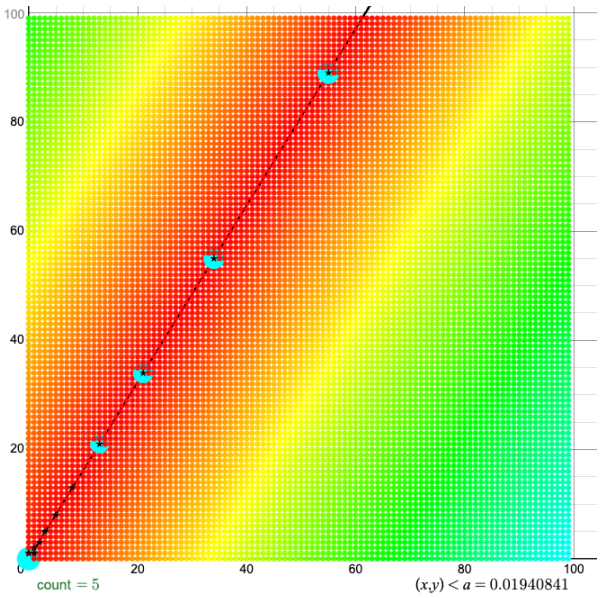



Figure 4: Graph of $y = \frac{1+\sqrt{5}}{2}x$, with closest points starred and in light blue

From the figure, the closest points (ordered by how close they are) are: $(55, 89), (34, 55), (21, 34)$, which are all points of the form $(F_n, F_{n+1})$, where $F$ denotes the Fibonacci sequence. Interestingly, we also see the Fibonacci sequence in computing successive convergents of the continued fraction for $\frac{1+\sqrt{5}}{2}$.

Note that

$$\frac{1+\sqrt{5}}{2} = [1;\overline{1}] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}$$

Thus, using a magic box yields

| | | 1 | 1 | 1 | 1 | 1 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 5 | 8 | ... |
| 1 | 0 | 1 | 1 | 2 | 3 | 5 | ... |

Table 1: The Magic Box for $\frac{1+\sqrt{5}}{2}$

And we notice successive convergents of $\frac{1+\sqrt{5}}{2}$ are $\frac{F_{n+1}}{F_n}$. In fact, the convergents will always be of this form, since $P_{n+1} = a_{n+1}P_n + P_{n-1} = P_n + P_{n-1}$, which is the same recursive definition as the Fibonacci sequence. Since the sequence begins with $1, 1$, this will generate all Fibonacci numbers. The same is true for the $Q$ terms, just one term staggered, hence the staggered fraction. This gives us a motivation to consider the relationship between convergents of a continued fraction and the best approximation for an irrational.

## §3.1 Convergents of Continued Fractions

**Example 3.3**

Consider $\sqrt{10}$ and its magic box:

| | | 3 | 6 | 6 | 6 | 6 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 19 | 117 | 721 | 4443 | ... |
| 1 | 0 | 1 | 6 | 37 | 228 | 1405 | ... |

Consider the convergents of $\sqrt{10}$: $\frac{3}{1}, \frac{19}{6}, \frac{117}{37}, \frac{721}{228}, \frac{4443}{1405}$. These fractions beg the question: Are they the closest possible rationals to $\sqrt{10}$

**Lemma 3.4**

For $n \geq 1$, the convergents satisfy $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

**Lemma 3.5**

The value of $\alpha$ is strictly between any two consecutive convergents, $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_n}{q_n}$.

*Proof.* Suppose $\alpha$ a positive irrational and $\frac{p_n}{q_n}$ the sequence of convergents.
We'd like to the show that $\forall n \geq 1$, that

$$|\alpha - \frac{p_n}{q_n}| < |\alpha - \frac{p_{n-1}}{q_{n-1}}|$$

We can express $\alpha$ as a finite continued fraction with an irrational last term $\alpha_n$
Let

$$a_n = a_n + \cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \ddots}}}$$

So

$$\alpha = [a_0; a_1, a_2, a_3, ..., a_{n-1}, \alpha_n]$$

Now note, from the recursive definitions

$$p_n = a_n p_{n-1} + p_{n-2}$$
$$q_n = a_n q_{n-1} + q_{n-2}$$

So

$$\alpha = \frac{p_n}{q_n} = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$$

Now consider

$$\alpha - \frac{p_{n-1}}{q_{n-1}}$$

Via algebra we show

$$\alpha - \frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-2}q_{n-1} - q_{n-2}p_{n-1}}{q_{n-1}(\alpha_n q_{n-1} + q_{n-2})}$$

Now recall from PSET #5 P2, that

$$p_{n-2}q_{n-1} - q_{n-2}p_{n-1} = (-1)^{k-2} = (-1)^k$$

Thus, the delta between $\alpha$ and $p_{n-1}/q_{n-1}$ is precisely.

$$\alpha - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_{n-1}(\alpha_n q_{n-1} + q_{n-2})} \tag{1}$$

Now we'd like to show that this delta in absolute value is strictly greater than the delta between $\alpha$ and the next convergent.

So

$$|\alpha - \frac{p_n}{q_n}| < |\alpha - \frac{p_{n-1}}{q_{n-1}}|$$

We'd like to manipulate this inequality such that we get a true statement $\forall n \geq 1$ Using equation (2)),

$$|\frac{(-1)^{n+1}}{q_n(\alpha_{n+1}q_n + q_{n-1})}| < |\frac{(-1)^n}{q_{n-1}(\alpha_n q_{n-1} + q_{n-2})}|$$

$$\frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} < \frac{1}{q_{n-1}(\alpha_n q_{n-1} + q_{n-2})}$$

$$\frac{1}{q_n(\alpha_{n+1}q_n + q_{n-1})} < \frac{1}{q_{n-1}(\alpha_n q_{n-1} + q_{n-2})}$$

Cross multiplying yields

$$q_n(\alpha_{n+1}q_n + q_{n-1}) > q_{n-1}(\alpha_n q_{n-1} + q_{n-2})$$

Note: $\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$

So

$$q_n(\alpha_{n+1}q_n + q_{n-1}) > q_{n-1}((a_n + \frac{1}{\alpha_{n+1}})q_{n-1} + q_{n-2})$$

$$\alpha_{n+1}q_n^2 + q_n q_{n-1} > q_{n-1}(a_n q_{n-1} + q_{n-2} + \frac{q_{n-1}}{\alpha_{n+1}})$$

Substituting in the recursive definition $a_n q_{n-1} + q_{n-2} = q_n$,

$$\alpha_{n+1}q_n^2 + q_n q_{n-1} > q_{n-1}(q_n + \frac{q_{n-1}}{\alpha_{n+1}})$$

$$\alpha_{n+1}q_n^2 + q_n q_{n-1} > q_{n-1}q_n + \frac{q_{n-1}^2}{\alpha_{n+1}}$$

$$\alpha_{n+1}q_n^2 > \frac{q_{n-1}^2}{\alpha_{n+1}}$$

$$\alpha_{n+1}^2 q_n^2 > q_{n-1}^2$$

Since $\alpha_n, q_n, q_{n-1}$ are positive for all $n \geq 1$, square rooting both sides yields

$$\alpha_{n+1}q_n > q_{n-1}$$

Also note: $\alpha_n > 1$, and $q_n$ is a strictly increasing sequence by the recursive definition, the above is true for all $n \geq 1$ since $\alpha_{n+1}q_n > q_n > q_{n-1}$. Thus, $\alpha_{n+1}q_n > q_{n-1}$. So

$$|\alpha - \frac{p_n}{q_n}| < |\alpha - \frac{p_{n-1}}{q_{n-1}}|$$

is true $\forall n \geq 1$

□

**Lemma 3.6**

The convergents get successively closer to $\alpha$: $|\alpha - p_n/q_n| < |\alpha - p_{n-1}/q_{n-1}|$.

*Proof.* Now we'd like to show any positive irrational $\alpha$ is strictly between consecutive convergents $p_{n-1}/q_{n-1}$ and $p_n/q_n$

Utilizing equation (2)), we see that for even $n$,

$$\alpha - \frac{p_{n-1}}{q_{n-1}} > 0 \implies \alpha > \frac{p_{n-1}}{q_{n-1}}$$

so for odd $n+1$,

$$\alpha - \frac{p_n}{q_n} < 0 \implies \alpha < \frac{p_n}{q_n}$$

So for even n,

$$\frac{p_{n-1}}{q_{n-1}} < \alpha < \frac{p_n}{q_n}$$

Similarly, for opposite parity,

$$\frac{p_n}{q_n} < \alpha < \frac{p_{n-1}}{q_{n-1}}$$

Thus $\alpha$ is strictly between consecutive convergents for all $n \geq 1$

□

**Theorem 3.7**

Let $\alpha$ be an irrational number and let $p_n/q_n$ be its $n$-th convergent for $n > 1$ (and specifically not $n \geq 1$). For all rationals $c/d$ with integers $c, d$ and $1 \leq d \leq q_n$, $|\alpha - c/d| \geq |\alpha - p_n/q_n|$ is satisfied.

Alternatively, the theorem can be stated as:

Let $\alpha$ be an irrational number and let $p_n/q_n$ be its $n$-th convergent for $n > 1$ (and specifically not $n \geq 1$). If $c/d$ is any rational number with integers $c, d$ such that $|\alpha - c/d| < |\alpha - p_n/q_n|$, then it must follow that $d > q_n$.

For this proof, we will the above 3 lemmas regarding the properties of convergents.

*Proof.* Suppose a rational number $c/d$ (in simplest form) that is a better approximation to $\alpha$ than $p_n/q_n$, such that

$$|\alpha - \frac{c}{d}| < |\alpha - \frac{p_n}{q_n}|$$

First, we consider the distance between $c/d$ and the convergent $p_{n-1}/q_{n-1}$:

$$\left|\frac{c}{d} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{cq_{n-1} - dp_{n-1}}{dq_{n-1}}\right| = \frac{|cq_{n-1} - dp_{n-1}|}{dq_{n-1}}$$

The numerator $|cq_{n-1} - dp_{n-1}|$ is an integer. It cannot be 0, because if it were, we would have $c/d = p_{n-1}/q_{n-1}$. This would imply, by our initial assumption, that $|\alpha - p_{n-1}/q_{n-1}| < |\alpha - p_n/q_n|$, which directly contradicts Lemma 3.6. Therefore, the numerator must be a non-zero integer, so $|cq_{n-1} - dp_{n-1}| \geq 1$. This gives us our first inequality:

$$\left|\frac{c}{d} - \frac{p_{n-1}}{q_{n-1}}\right| \geq \frac{1}{dq_{n-1}} \tag{2}$$

Next, we apply the triangle inequality:

$$\left|\frac{c}{d} - \frac{p_{n-1}}{q_{n-1}}\right| \leq |\alpha - \frac{c}{d}| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right|$$

Combining this with inequality (2), we get:

$$\frac{1}{dq_{n-1}} \leq \left|\alpha - \frac{c}{d}\right| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right|$$

Now, using our contradiction assumption that $|\alpha - c/d| < |\alpha - p_n/q_n|$, we can say:

$$\frac{1}{dq_{n-1}} < \left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| \tag{3}$$

By Lemma 3.5, $\alpha$ lies strictly between $p_{n-1}/q_{n-1}$ and $p_n/q_n$. This means that the sum of the distances between $\alpha$ and $p_{n-1}/q_{n-1}$ and $\alpha$ and $p_n/q_n$ is precisely equal to the distance between $p_{n-1}/q_{n-1}$ and $p_n/q_n$:

$$\left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right|$$

Using Lemma 3.6, we can simplify the right-hand side:

$$\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \left|\frac{p_nq_{n-1} - p_{n-1}q_n}{q_nq_{n-1}}\right| = \frac{|(-1)^{n-1}|}{q_nq_{n-1}} = \frac{1}{q_nq_{n-1}}$$

Substituting this result back into inequality (3):

$$\frac{1}{dq_{n-1}} < \frac{1}{q_nq_{n-1}}$$

Multiplying both sides by $q_{n-1}$ (which is positive) yields:

$$\frac{1}{d} < \frac{1}{q_n}$$

This implies $d > q_n$.

Thus, for a fraction $c/d$ with

$$|\alpha - \frac{c}{d}| < |\alpha - \frac{p_n}{q_n}|,$$

it must be true that $d > q_n$. □

# §4 Continued Fraction Generalizations

Now that we've shown continued fractions always generate "good approximations" for irrationals according to definition 3.2, we decided to explore some properties of continued fractions, specifically their periodicity.

**Theorem 4.1**
All irrational numbers with infinite periodic continued fractions are roots of quadratic equations with integer coefficients.

**Example 4.2**
$x = \frac{1+\sqrt{5}}{2} = [1;\overline{1}] \implies x^2 - x - 1 = 0$

$x = \sqrt{19} = [4;\overline{2,1,3,1,2,8}] \implies x^2 - 19 = 0$

$x = \frac{3+\sqrt{7}}{5} = [1;7,\overline{1,2,1,8,13,8}] \implies 25x^2 - 30x + 2 = 0$

*Proof.* Let $x$ be an irrational number with an infinite periodic continued fraction:

$$x = [a_0; a_1, a_2, \ldots, a_{m-1}, \overline{a_m, a_{m+1}, \ldots, a_{m+n-1}}]$$

where $a_i \in \mathbb{Z}$ for all $i$, $a_i > 0$ for $i \geq 1$.

Let $y$ be the purely periodic part of the continued fraction:

$$y = [\overline{a_m, a_{m+1}, \ldots, a_{m+n-1}}] = [a_m; a_{m+1}, \ldots, a_{m+n-1}, y]$$

Let $\frac{P_k}{Q_k}$, $P_k, Q_k \in \mathbb{Z}$, be the $k$-th convergent of the continued fraction $[a_m; a_{m+1}, \ldots, a_{m+n-1}]$.

By the recursive formula for $P_n$ and $Q_n$, we can write,

$$y = [a_m; a_{m+1}, \ldots, a_{m+n-1}, y] = \frac{yP_{n-1} + P_{n-2}}{yQ_{n-1} + Q_{n-2}}.$$

Rearranging we get,

$$Q_{n-1}y^2 + (Q_{n-2} - P_{n-1})y - P_{n-2} = 0.$$

Thus, $y$ is a root of a quadratic equation with integer coefficients.

Next, consider the continued fraction for $x$:

$$x = [a_0; a_1, a_2, \ldots, a_{m-1}, y].$$

Let $\frac{P'_k}{Q'_k}$ be the $k$-th convergent of $[a_0; a_1, a_2, \ldots, a_{m-1}]$. By the recursive formula for convergent fractions, we can write,

$$x = \frac{yP'_{m-1} + P'_{m-2}}{yQ'_{m-1} + Q'_{m-2}}.$$

Solving for $y$ from the equation for $x$ we get

$$y = \frac{P'_{m-2} - xQ'_{m-2}}{xQ'_{m-1} - P'_{m-1}}.$$

Since $y$ is the root of a quadratic equation, let $Ay^2 + By + C = 0$ for some $A, B, C \in \mathbb{Z}$. Substituting the expression for $y$ into this equation and simplifying gives,

$$\begin{aligned} &A(P'_{m-2} - xQ'_{m-2})^2 + B(P'_{m-2} - xQ'_{m-2})(xQ'_{m-1} - P'_{m-1}) \\ &\quad + C(xQ'_{m-1} - P'_{m-1})^2 = 0. \end{aligned}$$

Expanding this equation yields a polynomial in $x$ with degree at most 2.

Because $A, B, C, P'_{m-1}, P'_{m-2}, Q'_{m-1}, Q'_{m-2} \in \mathbb{Z}$, the coefficients of this polynomial will be integers. Therefore, $x$ is a root of a quadratic equation with integer coefficients. □

Our natural next step was to explore the converse:

**Claim 4.3 —** All irrational roots of quadratic equations with integer coefficients have infinite periodic continued fractions.

Let $y$ be an irrational root of a quadratic equation $Ax^2 + Bx + C = 0$, where $A, B, C \in \mathbb{Z}$ and $A \neq 0$. Since $y$ is irrational, the discriminant $D = B^2 - 4AC$ must be a positive non-square integer.

Without loss of generality, let $y$ be the root,

$$y = \frac{-B + \sqrt{D}}{2A} > 0.$$

**Definition 4.4.** Consider the continued fraction for $y = [a_0; a_1, a_2, \ldots]$. We define the sequence $\alpha_n$ as:

$$\begin{aligned} \alpha_0 &= [a_0; a_1, a_2, \ldots] = y \\ \alpha_1 &= [a_1; a_2, a_3, \ldots] \\ \alpha_2 &= [a_2; a_3, a_4, \ldots] \\ &\vdots \\ \alpha_n &= [a_n; a_{n+1}, a_{n+2}, \ldots]. \end{aligned}$$

**Lemma 4.5** (Recurrence Relation)
The terms of the sequence $\alpha_n$ satisfy the recurrence relation

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} \quad \text{with} \quad a_n = \lfloor \alpha_n \rfloor.$$

*Proof.* By definition,

$$\alpha_n = [a_n; a_{n+1}, a_{n+2}, \ldots] = a_n + \frac{1}{a_{n+1} + \frac{1}{a_{n+2}+\ldots}}.$$

Thus, we can write $\alpha_n$ as

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}.$$

Since $a_i \geq 1$ for $i \geq 1$, we know, $\alpha_i > a_i \geq 1$ for $i \geq 1$. Hence,

$$0 < \frac{1}{\alpha_{n+1}} < 1, \implies a_n = \lfloor \alpha_n \rfloor.$$

Rearranging, then gives the desired relationship,

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n},$$

for all $n \geq 0$. □

**Proposition 4.6**

For all $n \in \mathbb{N}$ where $n \geq 0$, the sequence $\alpha_n$ in the continued fraction expansion of an irrational quadratic root $y$ are of the form,

$$\alpha_n = \frac{c_n + \sqrt{D}}{d_n},$$

where $D = B^2 - 4AC$ is the discriminant of the quadratic equation, and $c_n, d_n$ are defined recursively by:

$$\begin{aligned} a_n &= \lfloor \alpha_n \rfloor \\ c_{n+1} &= a_n d_n - c_n \\ d_{n+1} &= \frac{D - c_{n+1}^2}{d_n} \end{aligned}$$

*Proof.* We prove this by induction on $n$.

**Base Case ($n = 0$):** By definition,

$$y = \frac{-B + \sqrt{D}}{2A}.$$

We can set $c_0 = -B$ and $d_0 = 2A$. Both $c_0$ and $d_0$ are integers (since $A, B \in \mathbb{Z}$). Thus,

$$\alpha_0 = y = \frac{c_0 + \sqrt{D}}{d_0},$$

which satisfies the required form.

**Inductive Step:** Assume that for some $n \geq 0$ and $c_n, d_n \in \mathbb{Z}$, the proposition is true:

$$\alpha_n = \frac{c_n + \sqrt{D}}{d_n}.$$

We want to show that $\alpha_{n+1}$ also follows this form and that $c_{n+1}, d_{n+1}$ are generated by the given recurrence relations.

From Lemma 4.5, we know $\alpha_{n+1} = \frac{1}{\alpha_n - a_n}$, where $a_n = \lfloor \alpha_n \rfloor$.

Substituting our inductive hypothesis for $\alpha_n$, we get,

$$\begin{aligned}\alpha_{n+1} &= \frac{1}{\frac{c_n+\sqrt{D}}{d_n} - a_n} \\ &= \frac{1}{\frac{c_n+\sqrt{D}-a_n d_n}{d_n}} \\ &= \frac{d_n}{\sqrt{D} - (a_n d_n - c_n)}\end{aligned}$$

Now, we define $c_{n+1} = a_n d_n - c_n$. Substituting this into the expression for $\alpha_{n+1}$ yields,

$$\begin{aligned}\alpha_{n+1} &= \frac{d_n}{\sqrt{D} - c_{n+1}} \\ &= \frac{d_n(\sqrt{D} + c_{n+1})}{(\sqrt{D} - c_{n+1})(\sqrt{D} + c_{n+1})} \\ &= \frac{d_n(c_{n+1} + \sqrt{D})}{D - c_{n+1}^2} \\ &= \frac{c_{n+1} + \sqrt{D}}{\frac{D - c_{n+1}^2}{d_n}}.\end{aligned}$$

Finally, we define

$$d_{n+1} = \frac{D - c_{n+1}^2}{d_n},$$

giving us,

$$\alpha_{n+1} = \frac{c_{n+1} + \sqrt{D}}{d_{n+1}},$$

completing the induction. Thus, we have shown that if $\alpha_n$ has the required form, then $\alpha_{n+1}$ also has the required form, with $c_{n+1}$ and $d_{n+1}$ generated by the claimed recurrence relations. □

**Lemma 4.7** (Integer Nature of Coefficients $c_n$ and $d_n$)
For all $n \in \mathbb{N}$ where $n \geq 0$, the coefficients $c_n$ and $d_n$ are integers.

*Proof.* We prove this by induction on $n$.

**Base Case ($n = 0$):** From the base case of Claim 4.6, $c_0 = -B$ and $d_0 = 2A$. Since $A, B \in \mathbb{Z}$, both $c_0$ and $d_0$ are integers.

**Inductive Step:** Assume that for some $n \geq 0$, $c_n \in \mathbb{Z}$ and $d_n \in \mathbb{Z}$. We want to show that $c_{n+1} \in \mathbb{Z}$ and $d_{n+1} \in \mathbb{Z}$.

1. $c_{n+1} \in \mathbb{Z}$: By definition, $a_n = \lfloor \alpha_n \rfloor$. Since $\alpha_n$ is a real number, its floor $a_n$ is an integer. By the inductive hypothesis, $c_n \in \mathbb{Z}$ and $d_n \in \mathbb{Z}$. Thus, $c_{n+1} = a_n d_n - c_n \in \mathbb{Z}$.

2. $d_{n+1} \in \mathbb{Z}$: Since $d_{n+1} = \frac{D - c_{n+1}^2}{d_n}$, we must show that $d_n \mid D - c_{n+1}^2$.

   We know by definition that,

   $$d_n = \frac{D - c_n^2}{d_{n-1}} \implies D - c_n^2 = d_n d_{n-1}.$$

   Substituting $c_{n+1} = a_n d_n - c_n$ into the expression for $D - c_{n+1}^2$ gives,

   $$\begin{aligned} D - c_{n+1}^2 &= D - (a_n d_n - c_n)^2 \\ &= D - (a_n^2 d_n^2 - 2a_n c_n d_n + c_n^2) \\ &= (D - c_n^2) - a_n^2 d_n^2 + 2a_n c_n d_n \end{aligned}$$

   Using the relation $D - c_n^2 = d_n d_{n-1}$, we can factor out $d_n$:

   $$D - c_{n+1}^2 = d_n d_{n-1} - a_n^2 d_n^2 + 2a_n c_n d_n = d_n(d_{n-1} - a_n^2 d_n + 2a_n c_n).$$

   Since $d_{n-1}, a_n, d_n, c_n$ are all integers (by inductive hypothesis and $a_n \in \mathbb{Z}$), the term $(d_{n-1} - a_n^2 d_n + 2a_n c_n)$ is an integer. Thus,

   $$d_n \mid D - c_{n+1}^2 \implies d_{n+1} = \frac{D - c_{n+1}^2}{d_n} \in \mathbb{Z}.$$

By induction, all $c_n$ and $d_n$ are integers for $n \geq 0$. □

Since all $c_n$ and $d_n$ are integers, we'd like to show that both are bounded by fixed values. Consider the conjugate of $\alpha_n$,

$$\overline{\alpha_n} = \frac{c_n - \sqrt{D}}{d_n}.$$

**Lemma 4.8** (Bounds on $c_n$ and $d_n$)
If $-1 < \overline{\alpha_n} < 0$, we can bound sequences $c_n$ and $d_n$ by the following:

$$0 < d_n < 2\sqrt{D}$$
$$0 < c_n < \sqrt{D}.$$

*Proof.* In the proof of Lemma 4.5, we've shown that $\alpha_n > 1$ for $n \geq 1$. Thus,

$$\alpha - \overline{\alpha_n} > 0 \implies \frac{\sqrt{D}}{d_n} > 0 \implies d_n > 0,$$

since $D$ is a fixed positive integer. Similarly, we know,

$$\alpha + \overline{\alpha_n} > 0 \implies \frac{2c_n}{d_n} > 0 \implies c_n > 0.$$

Additionally,

$$\overline{\alpha_n} < 0 \implies c_n < \sqrt{D}$$

$$\alpha > 1 \implies d_n < c_n + \sqrt{D} < 2\sqrt{D}.$$

Combining inequalities gives us the desired bounds,

$$0 < d_n < 2\sqrt{D}$$
$$0 < c_n < \sqrt{D}.$$

□

**Lemma 4.9**
If $-1 < \overline{\alpha_i} < 0$ for some $i \geq 0$, then $-1 < \overline{\alpha_n} < 0$ for all $n > i$.

*Proof.* We prove this by induction on $n > i$.

**Base Case** ($n = i$): By our assumption, the base case holds.

**Inductive Step:** Given that $-1 < \overline{\alpha_n} < 0$ for some $n \geq i$, we know from Lemma 4.5 that

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n} \quad \text{and} \quad \overline{\alpha_{n+1}} = \frac{1}{\overline{\alpha_n} - a_n}.$$

Since $a_n = \lfloor \alpha_n \rfloor \geq 1$ and $-1 < \overline{\alpha_n} < 0$, we get

$$-a_n - 1 < \overline{\alpha_n} - a_n < -a_n.$$

Thus,

$$\frac{1}{-a_n - 1} < \overline{\alpha_{n+1}} = \frac{1}{\overline{\alpha_n} - a_n} < \frac{1}{-a_n}.$$

Since $a_n \geq 1$, we have

$$-1 < \overline{\alpha_{n+1}} < 0.$$

Hence, by induction, $-1 < \overline{\alpha_n} < 0$ for all $n > i$. □

**Conjecture 4.10.** For all positive irrational roots of a quadratic equation $y$, there exists an $i \geq 0$ such that $-1 < \overline{\alpha_i} < 0$.

Assuming this conjecture holds, we know from Lemma 4.8 that the sequences $c_n$ and $d_n$ are bounded. Since both are integer sequences, there are only finitely many possible values for $(c_n, d_n)$ and thus only finitely many possible values of $\alpha_n$. By the Pigeonhole Principle, some pair must repeat. Hence, the continued fraction expansion of $y$ will become periodic from that point onward.

# §5 Buffet for Thought

This section of the paper is simply a collection of conjectures and other things to explore with continued fractions .

## §5.1 Definitions

First, consider some definitions of continued fractions

**Definition 5.1.** Let the $n$th **coefficient** of $\alpha$ be $a_n$ in its continued fraction expansion. $\alpha = [a_0; a_1, a_2, ..., a_n, ...]$

**Definition 5.2.** Define the **period** of the continued fraction of $\alpha$ to be the least such $k$ s.t.
$a_i = a_{i+k}$ for any coefficient $a_i$ of $\alpha$

**Definition 5.3.** Define $l(d)$ to be the period of the continued fraction of $\sqrt{d}$.

**Definition 5.4.** Define the **period sum** of irrational $\alpha$ where its period $n$ exists to be $\sum_{i=1}^{n} a_i$

**Definition 5.5.** Define $o(m)$ for all $\sqrt{m}$ in the form $[a_0; 1, 1, ..., 1, 1, 2a_0]$ to be the amount of 1's in this continued fraction expansion of $\sqrt{m}$. If $\sqrt{m}$ is not of this form, then $o(m) = 0$.

If $o(m) > 0$, then call $m$ a **repitend**.

## §5.2 Structure of coefficients

**Conjecture 5.1.** Palindromicity of $\sqrt{d}$
Suppose $\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_n}]$ for continued fraction expansion with a periodic part of length $n$. Then, for any $i, j \in \mathbb{N}$ such that $i + j = n$, the coefficients $a_i$ and $a_j$ satisfy $a_i = a_j$.

Some numericals to back this conjectures.

$$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$$

Notice how $2, 1, 3, 1, 2$ is palindromic.

$$\sqrt{57} = [7; \overline{1, 1, 4, 1, 1, 14}]$$
$$\sqrt{61} = [7; \overline{1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14}]$$
$$\sqrt{89} = [9; \overline{2, 3, 3, 2, 18}]$$
$$\sqrt{1377} = [37; \overline{9, 3, 1, 3, 1, 7, 2, 5, 4, 5, 2, 7, 1, 3, 1, 3, 9, 74}]$$

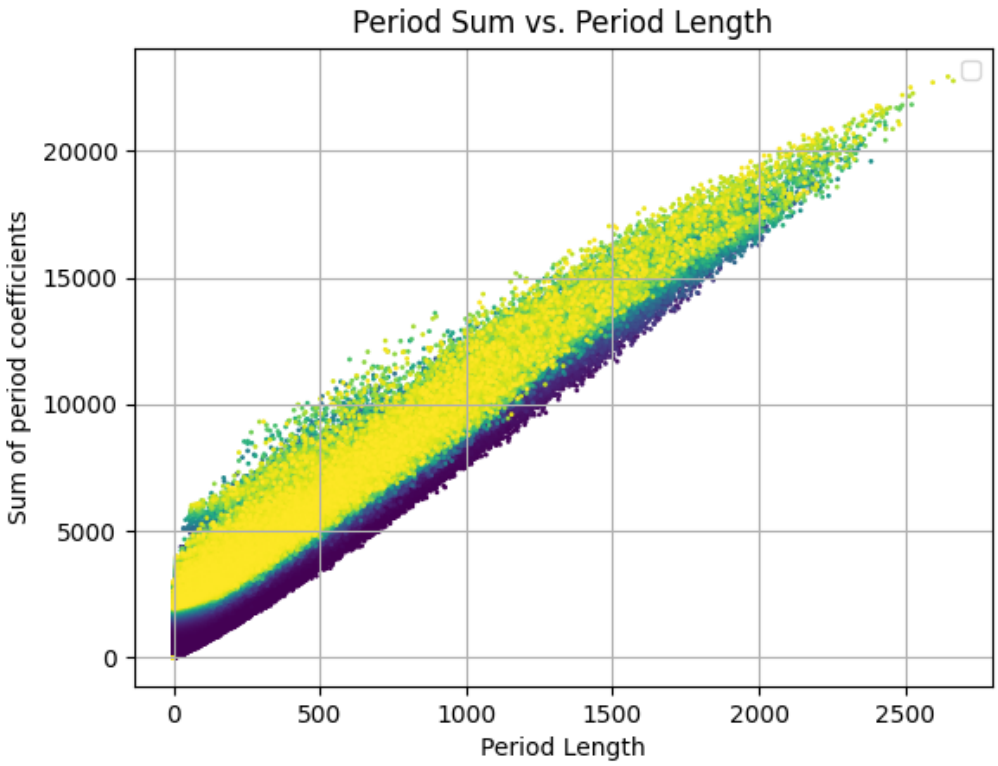
**Conjecture 5.2.** Last coefficient of $\sqrt{d}$
Suppose $\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_n}]$ for continued fraction expansion with a periodic part of length $n$. Then $a_n = 2a_0$

Again, the above numericals back this up.

## §5.3 Periods of $\sqrt{d}$

This subsection is mostly exploration with some conjectures.

**Example 5.3**

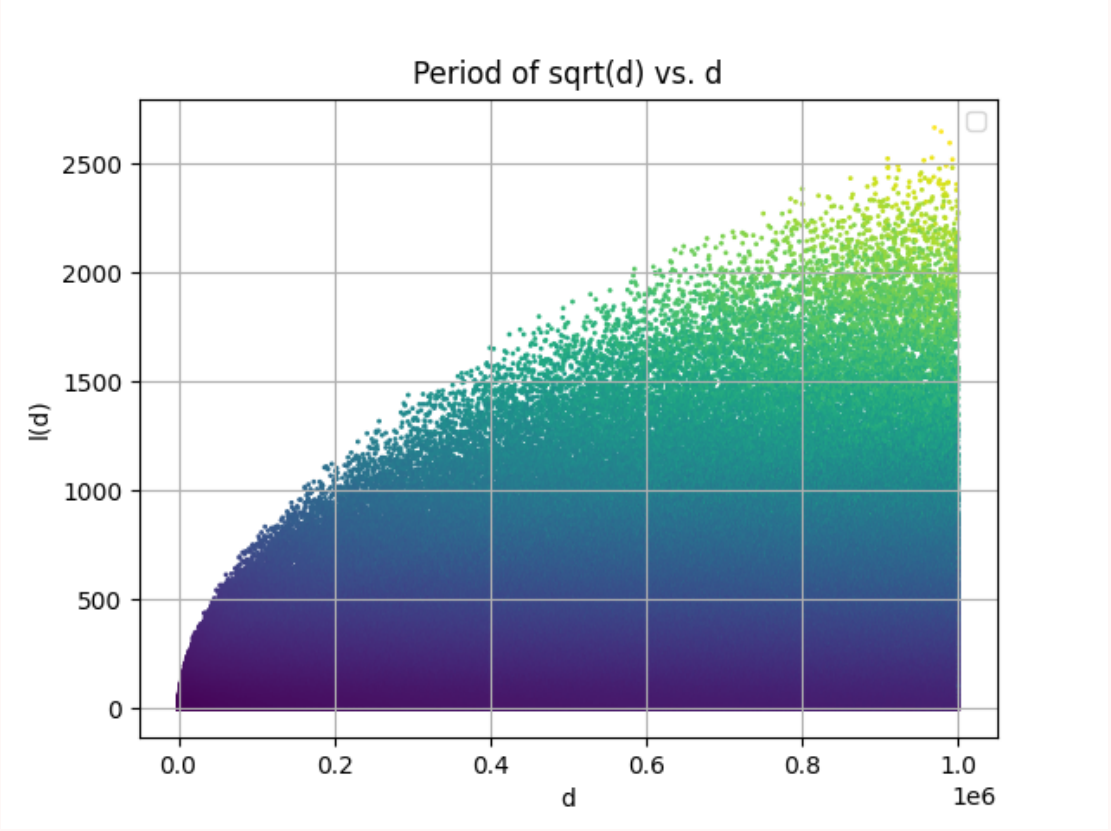



This scatter plot plots the period sum against the period length of $\sqrt{d}$ for $1 \leq d \leq 10^6$. The cooler colors represent lower values of $d$ where warmer colors represent the higher $d$ values.

Some interesting things to point are:

- There seems to be a clear lower bound on the ratio between period sum and period length. That lower bound of the ratio is approximately 6 for this range of $d$, however a better lower bound appears to be some (logarithmic) function of $d$.
- For $1 \leq n \leq d \leq 10^7$, the average ratio between period sum and period length is approximately $1.0295164675461108 \ln(0.9391029730042553n)$. (Which may or may not approach $ln(n)$ as the upper bound on the interval of $d$ tends to $\infty$
- Positive correlation with period sum and period length
- Observing the general direction from purple to blue to green to yellow, it seems that the sum of coefficients tends to increase at a faster rate (generally) than the period length.

**Example 5.4**




This plot graphs the period length of $\sqrt{d}$ against $1 \leq d \leq 10^6$. The cooler colors represent lower values of the period sum of $\sqrt{d}$, and warmer values for higher period sums of $\sqrt{d}$.

Some interesting things to point are:

- There also seems to be a strict upper bound on $l(d)$ based on $d$. $2\sqrt{d}$ holds as a good upper bound for a while, until $d = 1726$. Supposing $\exists$ differentiable $L(x)$ s.t. $L(d) \geq l(d) \forall d \in \mathbb{N}$, the plot suggests that $\lim_{x\to\infty} L'(x) \neq 0$, unlike some combination of an $n$th root function and a logarithm. But this is just an observation, $L(x)$ could be of the form $L(x) = a\sqrt[n]{x}\ln(bx)$ for $a, b, c \in \mathbb{R}$. Interesting that the graph suggests a definite (and non-trivial) upper bound.

**Conjecture 5.5.** Structure of $m$ given $l(m) \equiv 1 \mod 2$
If $l(m) \equiv 1 \mod 2$, then $m = 2^\alpha \prod_{p_i \equiv 1 \mod 4; p_i \text{ prime}} p_i^{e_i}$ for $\alpha \in \{0, 1\}$

## §5.4 Structures of specific $\sqrt{m}$

Consider $\sqrt{m} = [a_0; 1, 1, ..., 1, 1, 2a_0]$ with $o(m) = n$

**Conjecture 5.6.** $n \not\equiv 2 \mod 3$

**Conjecture 5.7.** Infinitely many such $m$ s.t. $o(m), o(m+1) > 0$ (twin repitend conjecture)

**Conjecture 5.8.** Forms of $m$.
If for $m \in \mathbb{N}$, $o(m) = n$ and $a_0 = \lfloor\sqrt{m}\rfloor$, then $m = a_0^2 + 1 + \frac{(2a_0+1)F_n}{F_{n+1}}$. ($F_n$ is the $n$th Fibonacci number.